**Chapter - 7**

**Surface and Volume Integrals**

**Surface Integrals**

Any integral which is to be evaluated over a surface is called a surface integral.

Suppose S is a piecewise smooth two-sided surface and is of finite area. Let (x, y, z) be a scalar valued continuous point function defined over S. Subdivide the surface S into n sub-surfaces

S1, S2,…….,Sn.

In each Sk, consider an arbitrary point Pk(xk, yk, zk) and form the sum

 

The limit of this sum as  in such a way that the largest dimension of each . This limit, if exists, is called the surface integral of (x, y, z) over S and is denoted as  .

Suppose S is a piecewise smooth two-sided surface and is of finite area. A unit normal  to any point P of the positive side (outer side) of S is called a positive or outward drawn unit normal.

Let (x, y, z) be a vector valued continuous point function defined over S. Then  is the normal component of  at P. Its integral over S is  and is known as the flux of  over S.

Other surface integrals are  where  .

**Result :** Let a continuous vector field  be defined on a surface S. If the surface S has projection Rxy on the xy plane, whose normal meets S in one and only one point, then prove:

  , provided .

**Remarks :**

1. If Ryz is the projection of S on the yz-plane, then

 

2. If Rzx is the projection of S on the zx-plane, then

 

**Solved examples**

**Example 1.** Evaluate where and S is the part of the plane which is located in the first octant.

**Solution :** Given that

and the surface S is given by

⟹ so

Hence unit normal to the surface S is

⟹

The surface S is the plane in the first octant, which in intercept form is given by

are the intercepts made by the plane on x, y and z axis respectively.

We take the projection R of surface S on the plane.

In plane and varies from.

Thus the region R is

Now,

**Example 2.** Evaluate where and S is the part of the plane which is located in the first octant.

**Solution :** Given that

and the surface S is given by

⟹ so

Hence unit normal to the surface S is

⟹

The surface S is the plane in the first octant, which in intercept form is given by

Here 6, 4, 2 are the intercepts made by the plane on x, y and z axis respectively.

We take the projection R of surface S on the plane.

In plane and varies from.

Thus the region R is

Now,

**Example 3.** Evaluate where and S is the surface of the plane in the first octant.

**Solution :** Given that

and the surface S is given by .

Hence unit normal to the surface S is

 and

We take the projection R of S in the first octant on the plane.

In plane and varies from.

Thus the region R is .

Now,

 .

**Example 4.** Evaluate where and S is the surface of cylinder included in the first octant between

**Solution :** Given that

and S is the surface of cylinder included in the first octant between

i.e. the surface ACDF as shown in fig.

Here S:

 G D

 F E

 O C

 and A B

 .

Hence the projection of S on the plane will not work.

We take the projection of S on plane. Let R be the projection.

Then R = rectangular region OCDG.

We have

and

In plane varies from and varies from

Now,

**Example 5.** Evaluate where and the surface S is the cube bounded by the planes

**Solution :** Given that

and the surface S consists of six surfaces

i.e. six planes of a cube.

They are shown in Fig. as

ABCD, OEFG, BEFC, AOGD,

 OABE and GDCF.

Then

(1) For the face ABCD:

 and

Then

(2) For the face OEFG:

 and

Then

(3) For the face BEFC:

 and

Then

(4) For the face AOGD:

 and .

Then

(5) For the face OABE:

 and

Then

(6) For the face GDCF:

 and .

Then

Now,

**Example 6.** Evaluate if and S is the surface of

 bounded by

**Solution.** Given that

and the surface S is given by

 .

Hence unit normal to the surface S is

 and

And

We take the projection R of surface S on plane.

Region R is bounded by .

Now,

**Example 7.** Evaluate if and S is the surface of bounded by

**Solution :** Given that

and the surface S is given by

 .

Hence unit normal to the surface S is

 and

And

We take the projection R of surface S on plane.

Region R is bounded by .

Now,

**Volume Integral**

Any integral which is to be evaluated over a volume is called a volume integral.

Suppose V is a volume enclosed in a closed surface S. Let (x, y, z) be a scalar valued continuous point function defined over V. Subdivide the volume V into n elements of volume V1, V2,……., Vn. In each Vk, consider an arbitrary point Pk (xk, yk, zk) and form the sum



The limit of this sum as  in such a way that the largest dimension of each. This limit, if exists, is called the volume integral or space integral of (x, y, z) over V and is denoted as 

If (x, y, z) be a vector valued continuous point function defined over V, then  is also an example of volume integral.

**Solved examples**

**Example 1.** Let Evaluate (a)(b)*,*

where V is the closed region bounded by the planes .

**Solution :** To find the limits of integration.

Consider the equation of a plane. ….. (1)

Putting Hence varies from

Putting Hence y varies from

and . Hence z varies from

**(b)**

**Example 2.** Evaluate , where and is the region bounded by the surfaces

**Solution :** Given that

and is the region bounded by the surfaces

**Example 3.** Let Evaluate, where and is the region bounded by the surfaces

**Solution :** Given that

and is the region bounded by the surfaces

Hence

**Example 4.** Evaluate where V is the closed region bounded by the cylinder

 and the planes .

**Solution :** Here the limits of integration are

**Example 5.** If then evaluate where V is the closed region bounded by the planes and

**Solution :** Here the limits of integration are

**Example 6.** If then evaluate

where V is the closed region bounded by the planes and

**Solution :** Here the limits of integration are

Now,

**Example 7.** Find the volume of the region common to the intersecting cylinders

 .

**Solution :** By symmetry the volume of the common region will be shared equally by eight octants. Hence the required volume V is given by

V = 8 (Volume of the common region in the first octant)

The limits of integration in the first octants are

 .

Now,

**7.4. Stoke's Theorem**

Suppose S is an open, two sided surface bounded by closed non-intersecting curve C (simple closed curve curve) traversed in the positive direction, and is a vector function of position with continuous derivatives. Then

 

**Solved examples**

**Example 1.** Verify Stokes theorem when , where S is the upper half surface of the sphere and C is its boundary.

**Solution :** By Stokes theorem we have

 ….. (1)

Here and

and

 ,

LHS of (1)

R = Projection of S on the xy plane

 LHS of (1)area of the unit circle ….. (2)

Now, C is the boundary of the upper half of the sphere, which is the circle in the xy plane, where and .

We take the parametric equation of C as .

On the xy plane, .

 .

Thus the stoke’s theorem is verified.

**Example 2.** Evaluate by Stoke’s theorem, where C is the curve .

**Solution :** By Stokes theorem, we have

 ….. (1)

Here ,

.

**Example 3.** By using Stoke’s theorem, evaluate , where C is the boundary of the surface S which is the upper half surface of the sphere

**Solution :** By Stokes theorem we have

 ….. (1)

Here ⟹

and ,

⟹

and

∴

and

Then equation (1) becomes

Here R is the Projection of S on the xy plane then varies from and varies from , then above equation becomes

 (∵ The function in 1st integral is odd and in 2nd integral is even)

**Example 4.** Prove that a necessary and sufficient condition that for every closed curve C is that  identically.

**Solution :** By Stokes theorem, we have

 ….. (1)

**Necessary condition :** Let for every closed curve C.

We have to show that .

We apply the method of contradiction.

Assume the contrary that  at some point P of S.

Since is continuously differentiable,  is continuous. Hence there will be some region containing the point P, where .

Suppose that S is the surface in this region and is the unit normal to it such that it is parallel to  at each point of S.

 is a positive constant.

If C is the boundary of S, Stoke’s theorem gives

But this contradicts the hypothesis that .

Hence the initial assumption must be wrong.

Hence  at all points of S.

**Sufficient condition :** Assume that 

 .

Then by (1)

**Divergence Theorem of Gauss**

Suppose V is the volume bounded by a closed surface S and is a vector function of position with continuous derivatives. Then



where is the positive (outward drawn) normal to S.

**Solved examples**

**Example 1.** By using Gauss divergence theorem, evaluate where and is the surface of the cube bounded by

**Solution :** By divergence theorem, we have

 ..... (1)

Here and is the surface of the cube bounded by

Now,

Then equation (1) becomes

**Example 2.** By using divergence theorem, evaluate where and is the surface of the cube bounded by

**Solution :** By divergence theorem, we have

 ….. (1)

 Here and is the surface of the cube bounded by

Now,

Then equation (1) becomes

**Example 3.** Verify divergence theorem for

taken over the rectangular parallelepiped .

**Solution :** By divergence theorem,

 ….. (1)

Here

]

 .

LHS of (1)

 ….. (2)

Now, S is the surface of a parallelepiped as shown in the fig.

 E D

 F G

 O C

 A B

RHS of (1)

 .

(1) For:

(2) For:

(3) For:

 .

(4) For:

(5) For:

.

(6) For:

With these values,

RHS of (1)

 .

Thus the divergence theorem is verified.

**Example 4.** Evaluate that, where and S is the surface of the solid cut off by the plane from the first octant.

**Solution :** We have 

By divergence theorem,

 .

**Example 5.** Show that, where V is the volume enclosed by a closed surface S.

**Solution :** We have .

 By divergence theorem,

**Example 6.** If then by divergence theorem, prove that, for any closed surface S.

**Solution :** By divergence theorem,

**Example 7.** Show that,

where S is the surface of the sphere .

**Solution :** Let .

We have .

By divergence theorem,

 .

 [∵

**Example 8.** If F and G be scalar functions with continuous derivatives of the second order at least, then show that

 .

**Solution :** By divergence theorem,

For

 ….. (1)

Interchange of F and G in (1) gives

 ….. (2)

Subtracting (2) from (1), we obtain

 (b)  (c) 

 (d) If and be scalar functions of position with continuous derivatives of the second order

 at least, then .

