**Chapter – 8**

**The Two-Body Central Force Problem**

**Equivalent One Body Problem**

**Theorem 1** : The problem of motion of two masses interacting only with one another always be reduced to a problem of the motion of a single mass.

**Proof**: Consider a conservative system of two mass points with position vectors respectively, where the only forces are those due to an interaction potential V.

Hence the potential V depends on the distance between the particles.

 If , ….. (1)

then

Such a system of two particles has six degrees of freedom and hence six independent generalized coordinates.

Let us choose these to be the three components of the radius vector to the centre of mass and the three components of the difference vector given by (1).

Lagrangian of the system is

 ….. (2)

Let be the position vector of the centre of mass of the system.

 ….. (3)

Multiplying equation (1) by

 ….. (4)

(3)(4)

Multiplying equation (1) by

 ….. (5)

(3) (5)

Substituting these values of and in equation (2) we get ,

 ….. (6)

It is seen that three coordinates of are cyclic. as L does not contain .

 Lagranges equations , becomes

 or constant

 0 or constant

 0 or constant

Hence, the centre of mass moves with constant velocity or is at rest, so that drop the first term from the lagrangian in equation (6), we get

or here is known as the reduced mass.

Now, the Lagrangian is exactly that of a single particle with mass at a distance from a fixed centre force.

Thus the central force motion of the two bodies can always be reduced to an equivalent one body problem.

**The Equations of Motion and First Integrals**

**Result 1:** In a central force field, the angular momentum of a particle remains constant.

**Proof :** We have as said earlier that in a central force field the position vector of a particle is always along impressed force

 i.e.,

i.e., the angular momentum vector is constant.

**Result 2:** The path of a particle in a central force field lies in one plane.

**Proof :** We know that the angular momentum of a particle remains constant during the motion in a central force field.

But

 is always perpendicular to , which has a fixed direction.

Therefore, the particle moves in space such that its radius vector always lies in a plane whose normal is fixed. This means that the motion of a particle is in a plane.

**Result 3:** In a central force field, the areal velocity is conserved.

The motion of a single particle in space is described by three coordinates.

By choosing the polar axis to be in the direction of angular momentum , the motion is always in the plane perpendicular to the polar axis.

 So consider the Lagrangian in polar coordinates

Clearly, here does not contain , i.e., is cyclic

One of the two equations of motion is then

 , where is the constant magnitude of the angular momentum.

The factor is inserted because is the areal velocity (the area swept out by the radius vector per unit time) denoted by or

Thus, Areal velocity .

 In a central force field, the areal velocity is conserved.

**Motion of a Single particle**

The conservation of angular momentum is equivalent to saying the areal velocity is constant which is well known **Kepler’s second law** states that “*The radius vector sweeps out equal areas in equal time*.”

The Lagrange equation for the coordinate r is

or where

But as areal velocity

or

which is a second order differential differential equation involving r only.

The total energy E is given by

Let at time t, r have initial value .

Take the integral of both sides of the equation from initial state to the state at time t.

i.e.,

It gives t as the function of r and the constants of integration

Again

or

Put so that , hence above gives

 , which is the equation of motion of particle moving in a central force field.

**Example.** Show that the equation of the orbit can be put in the form

of a particle moving in a central force field

**Solution :** The orbit of the particle moving in a central force field is given by

 ….. (1)

In case of a central force field F, the potential V is given by .

Take as the initial value of , relation (1) becomes

Put ….. (2)

 where

or ….. (3)

Now,

Further take

Equation (3) which is the required orbit.

**Virial**

Consider a system of mass points with position vectors

Let be the applied force, on the ith particle, which includes the forces of constraints if any. The quantity called the *virial* or the *virial of Clausius* of the system.

The bar over a quantity denotes the time average of the quantity.

[Also note that means and etc.]

**Virial Theorem :**

**Statement :** For a system moving in a finite region of space with finite velocity, the time average of kinetic energy is equal to the virial of the system.

i.e.,

**Proof :** Consider a general system of mass points with position vectors and applied forces **(**including any forces of constraints).

The fundamental equations of motions are

 ….. (1)

 Let

Integrating both the sides w.r.t. ‘t’ from and dividing by , we get

or

Assume that the system executes a motion in a finite region with finite velocity then the quantity and hence will be bounded.

Hence is bounded.

 is bounded.

Hence

**Note :** If the forces are derivable from a potential then .

In this case, virial theorem becomes .

**Theorem 2 :** If the potential energy is a homogeneous function of degree in the radius vector , then the motion of a conservative system takes place in a finite region of space only if the total energy is negative.

**Proof:** As the system is finite, the virial theorem gives

 ….. (1)

Since the system is conservative, we have

Then (1) becomes …..(2)

If V is a homogeneous function of degree in , then by Euler’s theorem for homogeneous function, we have

 using (2)

or

or

i.e.,

i.e., The total energy is negative, where .

**Theorem 3 :** For a particle moving under a central force such that , the virial theorem reduces to .

**Proof:** From Virial theorem, we have

Since the system is conservative, we have

For a single particle, above reduces to

Now,

**Example**: Consider a system in which the total forces acting on the particles consist of conservative forces and frictional forces proportional to the velocity. Show that for such a system the virial theorem holds in the form providing the motion reaches a steady state and is not allowed to die down as a result of the frictional forces.

**Solution :** Let be the total force on the ith particle of the system.

 ….. (1)

Now

 where total force)

 as =

Taking time average, we get

 ….. (2)

Now being finite, the quantities and are bounded and hence their time averages must vanish.

Then (2)

 .

**Differential Equation for the Orbit**

**Theorem 4 :** For a central force field F, the differential equation for the orbit is given by

**Proof :** A central force motion is a motion in a plane.

Hence the Lagrangian in plane polar coordinates is

 ….. (1)

As is cyclic coordinate, .

 Lagrange’s equation becomes

 , is the constant magnitude of the angular momentum.

Lagrange’s r-equation is

or ….. (2)

Now, then

and

Substituting the values of in equation (2), we get

 i.e.,

 .

**Example :** Show that if a particle describes a circular orbit under the influence of an attractive central force directed towards point on the circle, then the force varies as the inverse fifth power of the distance.

**Solution:** Let O be the centre of force and the particle be at at any time t. OX is the initial line. The particle describes a circle with centre C and radius c.

By cosine rule, we have , for .

In adjoining figure, for we have

or

Then the equation of the path

 , becomes

 , where negative sign indicates that the force is of attractive nature.

i.e.,

Hence the force F varies as the inverse fifth power of the distance.