**Chapter - 1**

**solid GEOMETRY**

**The Cone**

**Definition :** A **cone** is a surface generated by a straight line which passes through a fixed point and satisfies one more condition of the type that it may intersect a given curve or touch a given surface or make a given angle with a line through the fixed point.

The fixed point is called the **vertex** of the cone; the straight line which generates the cone is called the **generator** of the cone and given curve is called the **guiding curve**.

A

P

C

A

P

C

A ← vertex, AP ← generator, C ← guiding curve

Thus a cone is essentially a set of lines called generators through a given point. Alternatively, the cone is a set of points on its generators.

**The Right Circular Cone**

**Definition :** A **right circular cone** is a surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line through the fixed point.

The fixed point is called the **vertex**; the fixed line through the fixed point (vertex) is called the **axis** and the constant angle is called the **semi-vertical angle** of the cone.

**Remark :** Every section of a right circular cone by any plane perpendicular to its axis is a circle.

**Equation of a Right Circular Cone**

1. **General form**

**To find the equation of the right circular cone whose vertex is the point** (𝛂, 𝛃, 𝛄) **and whose axis is the line and semi-vertical angle** θ.

Let V be the vertex and, VA, the axis of the cone. The equations of axis VA of the cone are given by

θ

P(x,y,z)

V(α,β,γ)

A

∴ The d.r.’s of the axis VA are *l*, m, n.

Let P (x, y, z) be any point on the cone. Then VP is the generator of

the cone and its d.r.’s are x – α, y – β, z –γ .

The semi-vertical angle is θ, i.e., an angle between VA and VP is θ.

∴ cos θ =

⟹ []

=

This is the required equation of the right circular cone.

1. **Standard form**

**To find the equation of the right circular cone whose vertex is at the origin, axis along Z-axis and semi-vertical angle** θ.

P(x,y,z)

Z

θ

Y

O

X

Let P (x, y, z) be any point on the cone whose vertex is

at the origin O (0, 0, 0).

Therefore the d.r.’s of the generator OP are

x – 0, y – 0, z – 0, i.e., x, y, z.

Z - axis, whose d.r.’s are 0, 0, 1, is the axis of the cone.

If θ is the semi-vertical angle (angle between Z-axis and OP),

then cos θ = =

⟹ () =

⟹ =

or = (

⟹ = .

This is the equation of the right circular cone whose vertex is at the origin, axis along z-axis and semi-vertical angle θ.

**Remark :** Equation of the right circular cone whose vertex is at the origin, axis along X-axis (or Y-axis) and semi-vertical angle θ is = (or = ).

**Solved Examples**

**Example 1.** Find the equation of right circular cone whose vertex is at the origin, whose axis is the line x = y = z and which has semi-vertical angle .

**Solution :** Axis of the cone is the line x = y = z , i.e.,

∴ The d.r.’s of the axis of the cone are 1, 2, 3.

Vertex of the cone is the origin O (0, 0, 0).

Let P (x, y, z) be any point on the cone. Then the d.r.’s of the generator OP are

x – 0, y – 0, z – 0, i.e., x, y, z.

Now, the semi-vertical angle (angle between the axis of the cone and generator OP) is .

∴ cos =

⟹ =

⟹ 42 () = 4 (

⟹ 21 () = 2 + 12 y z + 6 z x)

⟹ 24 y z 12 z x = 0.

This is the required equation of the right circular cone.

**Example 2.** Show that the equation of the right circular cone with vertex (2, 3, 1), axis parallel to the line and one of its generators having direction cosines proportional to

is 12 y z + 6 z x + 36 y + 22 z 19 = 0.

**Solution :** Vertex of the right circular cone is (2, 3, 1).

Axis is parallel to the line , i.e., .

⟹ The d.r.’s of the axis are .

One of the generators of right circular cone is having direction cosines proportional to .

⟹ The d.r.’s of one of the generators are .

If P (x, y, z) be any point on the cone, then the d.r.’s of the generator passing through this point are x – 2, y – 3, z – 1.

Since the cone is right circular, the generators make a constant angle with the axis.

⟹ =

⟹ =

⟹

⟹

⟹

⟹ 12 y z + 6 z x + 36 y + 22 z 19 = 0.

This is the required equation of the right circular cone.

**Example 3.** Prove that the semi-vertical angle of a right circular cone admitting sets of three mutually perpendicular generators is.

**Solution :** If the vertex of the cone is origin, axis of the cone is z-axis and semi-vertical angle θ, then the equation of the cone is

or ….. (1)

If the cone has three mutually perpendicular generators, then a + b + c = 0,

where a, b, c are the coefficients of x2, y2, z2 respectively in the equation of the cone.

(By the condition that the cone has the sets of three mutually perpendicular generators)

⟹ 1 + 1 − = 0 (∵ In (1), a = 1, b = 1, c = )

⟹ tanθ = or θ = .

**Example 4.** Find the equation of right circular cone whose vertex is (2, –3, 5), axis makes equal angles with the co-ordinate axes and semi-vertical angle is .

**Solution :** Since the axis of right circular cone makes equal angles with the co-ordinate axes, its d.r.’s are 1, 1, 1 and hence its equation is

(∵ ( 2, –3, 5 ) is the vertex )

Let P (x, y, z) be any point on the cone. Then the d.r.’s of the generator through P are

x – 2, y + 3, z – 5.

Now, the semi-vertical angle is , and hence

cos = 

⟹ =

⟹ 9 [ ] – 4 [ ] 2 = 0

⟹ 5 [] − 8 ( x y + y z + z x ) − 4 x + 86 y − 58 z + 278 = 0.

**Example 5.** Find the equation of right circular cone generated by revolving a straight line

5 y + 2 z = 10 , x = 0 about the Z-axis.

**Solution :** Generating line for the right circular cone is

5 y + 2 z = 10 , x = 0 ….. (1)

The vertex V of the cone is the point of intersection of (1) and the Z-axis.

Equation of the Z-axis is x = 0, y = 0 ….. (2)

Solving (1) and (2), we get x = 0, y = 0, z = 5 and hence the vertex is V (0, 0, 5).

Let P (x, y, z) be any point on the cone and θ be the semi-vertical angle. Therefore the d.r.’s of VP are x – 0, y – 0, z – 5 and of the axis (Z –axis) are 0, 0, 1.

∴ cos = = ….. (3)

Rewriting (1),

The d.r.’s of the generating line are 0, 2, 5.

Since the angle between generating line and the Z-axis is θ, we have

cos = = ….. (4)

(3) and (4) ⟹ =

⟹ 25 [] = 29

or = 0

This is the required equation of the right circular cone.

**The Cylinder**

**Definition :** A cylinder is the surface generated by a straight line (variable line) which is parallel to a fixed straight line and satisfying one other condition of the type that it intersects a given curve or touches a given surface.

The fixed straight line is called the **axis of the cylinder**, the variable line is called **a generator** and if the generator intersects the given curve, the curve is called a **guiding curve.**

Axis

Generators

Guiding cucurve

Circle

Guiding curves

**Right Circular Cylinder (RCC)**

**Definition :** A **right circular cylinder** is the surface generated by a straight line which is parallel to a fixed line and is at a constant distance from it.

The fixed straight line is called the **axis** and the constant distance is the **radius** of the cylinder.

A right circular cylinder may also be defined as a cylinder whose guiding curve is a fixed circle and whose generator is normal to the plane of the circle. The radius of the circle is the radius of the cylinder.

The axis of the cylinder passes through the centre of the circle and is also perpendicular to the plane of the circle.

**Equation of a Right Circular Cylinder**

**(A) General Form:**

**To find the equation of the right circular cylinder whose radius is r and axis is the line**

**.**

Suppose that P (α, β, γ) is any point on the right circular cylinder and A (x’, y’, z’) is the fixed point on the axis and r is the radius.

Draw PQ perpendicular to the axis BC.

From the right angled triangle AQP, we have

AP2 = AQ2 + QP2 = AQ2 + r2 ….. (1)

Now,

and AQ = Projection of AP on the axis BC.

A(x’,y’,z’)

r

Q

P(α,β,γ)

B

C

=

(by projection formula)

∴ AQ2 =

∴ Equation (1) becomes

=

Then the locus of the point P (α, β, γ) is

This is the required equation of the cylinder.

**Remarks:**

(1) The equation of right circular cylinder can also be written as

….. (∗)

which is easy to remember.

(2) We can find the equation of right circular cylinder if we know

(i) its radius, (ii) d.c.’s of axis (iii) one point on the axis .

**(B) Particular Cases:**

**Case (1):** Let the axis be

In this case, x’ = y’ = z’ = 0.

∴ The equation of right circular cylinder becomes,

(*l*2 + m2 + n2) (x2 + y2 + z2 – r2) − (*l*x + my + nz)2 = 0

**Case (2):** If the axis is the z-axis. Then *l* = m = 0, n = 1.

Also, (x’, y’, z’) = (0, 0, 0) is the point on the z-axis

∴ The equation (∗) gives x2 + y2 = r2.

**SOLVED EXAMPLES**

**Example 1.** Find the equation of the right circular cylinder of radius 2 whose axis pass through the point ( 1, 0, 3 ) and has d.c.’s proportional to ( 2, 3, 1 ).

OR

The axis of a right circular cylinder of radius 2 is ; show that its equation is:

.

**Solution :** Axis of the right circular cylinder is passing through the point ( 1, 0, 3 ) and has d.c.’s proportional to ( 2, 3, 1 ).

A(1,0,3)

2

Q

P(x,y,z)

∴ Its equation is .

Now, the equation of a right circular cylinder whose axis is

and radius r is

….. (1)

Here *l* = 2, m = 3, n = 1 and (x’, y’, z’) = (1, 0, 3)

∴ Equation (1) gives

⟹

⟹ .

This is the required equation of the right circular cylinder.

**Example 2.** Find the equation of the right circular cylinder of radius 2 whose axis is the line

.

**Solution :** Axis of the right circular cylinder is the line , and the radius is 2.

Now, the equation of a right circular cylinder with axis and the radius r is

….. (1)

Here *l* = 2, m = 1, n = 2 and ( x’, y’, z’) = (1, 2, 3)

∴ Equation (1) gives

⟹

⟹

This is the required equation of the right circular cylinder.

**Example 3.** Find the equation of the right circular cylinder which passes through the circle

x2 + y2 + z2 = 16, x + y + z = 4.

**Solution :** Here the guiding circle is

x2 + y2 + z2 = 16, x + y + z = 4.

Since the axis of the cylinder is perpendicular to the plane of the circle, its d.r.’s are the d.r.’s of the normal to the plane x + y + z = 4.

Hence the d.r.’s of the axis are 1, 1, 1.

Then the d.r.’s of the generator of the cylinder will be1, 1, 1 (∵ generator is parallel to the axis)

If P (α, β, γ) is any point on the cylinder, the equation of the generator through P is

Any point on this generator is ( .

As generator lies on the cylinder and passes through the guiding curve (here it is a circle), this point lies on the given circle.

∴ ,

⟹ , and

⟹

or 2(

or .

Generalizing α, β, γ, the required equation of the cylinder is

x2 + y2 + z2 – xy – yz – zx – 16 =0.

**Example 4.** Prove that the equation of the right circular cylinder described on the circle through the points A (1, 0, 0), B (0, 1, 0) and C (0, 0, 1) as the guiding curve is

x2 + y2 + z2 − yz – zx – xy – 1 = 0.

**Solution :** The circle through the points A, B, C is the section of the sphere passing through O, A, B, C by the plane through A, B, C.

The general equation of the sphere is

x2 + y2 + z2 + 2ux + 2vy + 2wz + d =0

If the sphere passes through O ( 0, 0, 0 ) then its equation is

x2 + y2 + z2 + 2 u x + 2 v y + 2 w z = 0.

Also, if it passes through A (1, 0, 0), B (0, 1, 0), C (0, 0, 1), then we have

1 + 2 u = 0, 1 + 2 v = 0, 1 + 2 w = 0

⟹ u = v = w =

∴ The equation of the sphere is

x2 + y2 + z2 – x – y – z = 0

The plane ABC is

, i.e., x + y + z = 1 (Intercept form)

Hence the guiding circle is given by

x2 + y2 + z2 − x – y – z = 0, x + y + z = 1

The axis of the cylinder is perpendicular to the plane of the circle and hence its d.r.’s are 1, 1, 1.

(∵ d.r.’s of the plane x + y + z = 1 are 1, 1, 1)

∴ The d.r.’s of the generator are 1, 1, 1.

Let P (α, β, γ) be any point on the cylinder, then the equation of the generator through P is

(say)

Any point on it is (.

If this point lies on the circle then

and

i.e. 3k2 + 2k (α + β + γ) + α2 + β2 + γ2 – 1 = 0 and 3k + α + β + γ = 1

⟹ 3α2 + β2 + γ2 – 1= 0

⟹ (α2 + β2 + γ2) − (

∴ The required equation of the cylinder is

x2 + y2 + z2 – yz – zx – xy – 1 = 0.