**Chapter – 7**

**Elementary Principles of Classical Mechanics**

**Some Definitions**

**(1) Velocity :** If is a position vector (or radius vector) of a particle from some given origin (fixed point), then the velocity of a particle isgiven by

.

**(2) Acceleration :** If is the velocity of the particle at any instant then its acceleration is given by

 .

**(3) Linear momentum :** The linear momentum of a particle, denoted by , is defined as the product of mass of a particle and its velocity,

i.e., , where is the mass of a particle and is its velocity.

Linear momentum of a particle is directed in the same direction as velocity .

**(4)** **Angular momentum :** The angular momentum of a particle about any fixed point , denoted by , is defined as

 ,

where is the radius vector from to the particle and is its linear momentum.

The angular momentum may also be considered as the **moment of linear momentum**.

**(5)** **Torque or Moment of a force :** The moment of a force or torque about point , denoted by , is defined as

 , where is the force acting on a particle.

**(6) Work :** If a particle having position vector is displaced through a distance due to the application of force , thenthe work done by the force upon the particle, denoted by , is given by

If the particle is displaced from point 1 to point 2, then the work done by the force upon the particle is

given by

 or

**(7) Virtual displacement :** Any real displacementis always associated with a change in time. The displacement which takes place without any change in time, i.e., the displacement takes place instantaneously, is called a virtual displacement.

Thus, the virtual displacement is the imaginary or infinitesimal displacement.

We denote the real displacement by and the virtual displacement by .

**(8) Virtual work:** Work done by a force on the particle when a virtual displacement is given to the particle is called virtual work.

If is a virtual displacement of a particle, then the virtual work done by the force is give by

**(9) Kinetic energy :** The kinetic energy of a particle is defined as half the product of the mass of a particle and the square of its velocity.

If a particle of mass is moving with velocity , then its kinetic energy, T, is given by

Kinetic energy is the scalar quantity.

**Note:**

**(10) Conservative force :** If the work done by a force in moving a particle from one point to another is independent of the path traversed by the particle, then the force (and the system) is said to be conservative.

**Conservation Principles (Laws)**

The word ‘**conservation**’ applies in the sense of **constantness** when some characteristic of the motion of a system remains constant in time. There are conservation laws relating to linear momentum, angular momentum, energy and various other quantities.

**Theorem 1. (Conservation theorem for the linear momentum of a particle):** If the total force , acting on a particle is zero then and the linear momentum is conserved.

**Proof :** Let the total force acting on a particle is zero, i.e., .

By Newton’s second law of motion, , where is a linear momentum of a particle.

 ,

 a constant vector, i.e., the linear momentum of a particle is conserved.

**Theorem 2. (Conservation theorem for the angular momentum of a particle):** If the total torque, , on the particle is zero, then and the angular momentum is conserved.

**Proof:** By definition of angular momentum and torque,

 and

 [ and ]

 ( ) [ ]

 [ ) ]

 [ ) ] [ ]

 If the total torque , on the particle is zero, then .

 a constant vector, i.e., the angular momentum of a particle is conserved.

**Theorem 3. (Conservation theorem for the linear momentum of a system of particles) :** If the total external force on the system of particles is zero, the total linear momentum is conserved.

**Proof :** Consider the system consisting of particles. Let be the mass of the th particle and be its position vector.

By Newton’s second law, for th particle, we have

or

Hence for the system of particles,

⟹ [ ]

or [ , the total external force acting on the system ]

or ….. (1)

where is the total linear momentum of the system.

If the total external force on the system of particles is zero, i.e., , then from (1), we get

 a constant vector

 The total linear momentum is conserved.

**SOLVED EXAMPLES**

**Example .** Show that for a single particle with constant mass the equation of motion implies the following differential equation for the K.E., , while if the mass varies with time the corresponding equation is .

**Solution :** **Case I:** is constant

Consider a particle of mass moving with velocity .

 By Newton’s second law of motion,

Now, the K. E.,

**Case II:** is variable

Linear momentum of the particle is and K. E., . Then

**Constraints**

Constraints mean limitations or restrictions. Constrained motion means restricted motion. In many situations, the object in motion is restricted or constrained to move in such a way that its coordinates and/or velocity components must satisfy some prescribed relations at every instant of time. These relations can be expressed in the form of either equations or inequalities.

For example,

1. Motion of a billiard ball on the table. Its motion is restricted by the boundaries of the table, and it moves on the surface of the table. Thus, the motion of a billiard ball on a billiard table is a constrained motion.
2. In rigid bodies, the motion must be such that the distance between any two particles is always the same.
3. The beads of an abacus are constrained to one dimensional motion by the supporting wires.
4. Gas molecules within a container are constrained by the walls of the vessel to move only inside the container.

**Degrees of Freedom**

The number of independent quantities required to specify the position of the system completely is called the **degrees of freedom** of the system.

Examples:

(1) A particle moving in a plane has two degrees of freedom because its position can be given by two cartesian coordinates or two polar coordinates .

(2) A system consisting of particles moving freely in space requires coordinates to specify its position. Thus the number of degrees of freedom is .

**Generalized Coordinates and Generalized Velocities:**

Any quantities which completely determine the position of a system with degrees of freedom are called **generalized coordinates** of the system.

Generally they are denoted by or , .

The quantities are called **generalised velocities**.

In above example (1), or are generalized coordinates and hence the generalized velocities are or .

**Note:** The generalized coordinates alone do not determine the state (mechanical) of the system. It is essential to know the coordinates and velocities simultaneously.

**Principle of Virtual Work:**

Consider a system of particles with position vectors. The virtual displacement of a particle is the displacement in no time or in time .

Hence the virtual displacement is independent of time .

Let the system be in equilibrium. Then each particle of the system is in equilibrium,

i.e., the total force on each particle vanishes, .

Then the virtual work of the force in the displacement vanishes,

i.e., , .

∴ The condition for equilibrium of a system is that the virtual work of the applied forces vanishes,

i.e., .

This is called the **principle of virtual work**.

**D’Alembert’s Principle**

The virtual work on a mechanical system (for which the net virtual work of the forces of constraints vanishes) by the applied forces and the reversed effective forces is zero,

i.e., ,

where denotes the applied force on the th particle of the system.

**Proof :** Consider a system consisting of particles whose position vectors are given by .

The equations of motion of the system are

 or ,

i.e., ….. (1)

Eq. (1) each of the particles of the system is in equilibrium under the forces and . Hence the system of the particles is in equilibrium.

 The sum of the virtual work of and is zero,

i.e.,

⟹ ….. (2)

Now, ,

where & are the applied force and force of constraints respectively on th particle of the system.

 ….. (3)

If we restrict ourselves to the systems in which net virtual work of the forces of constraint vanishes then

 ….. (4)

(3) & (4) ⟹

**Lagrange’s Equations of Motion**

Lagrange’s Equations of motion for conservative system are

 ,

**Derivation of the Lagrange’s equations of motion for conservative system in the form**

 **, from D’Alembert’s principle**

Let the system of particles be specified by generalised coordinates .

Consider that the constraints are holonomic. Then the position vectors of the particles are expressed as

 ….. (1)

D’Alembert’s principle is

or ….. (2)

[ dropping the superscript in ]

From (1), we have

 ….. (3) []

and or

 ….. (4)

Now,

 ….. (5)

where ….. (6)

are called the components of generalized forces (or in short generalised forces).

Also, [ using (3) ]

 =

 [∵ K. E. of the thparticle, ]

 ….. (7)

where is the total K. E. of the system.

Using (5) and (7) in (2), we get

 [ The constraints are holonomic, all are independent ]

or , ….. (8)

Eqns. in (8) are called Lagrange’s equations of motion. They are partial differential equations of 2nd order.

For a conservative system, we have

 ….. (9)

 ….. (10)

The P.E. is independent of and hence

 ….. (11)

Using (10) and (11) in (8), we have

or , where .

The quantity L is called the **Lagrangian** of the system.

**SOLVED EXAMPLES**

**Example 1.** Construct the Lagrangian for a particle moving in space and then deduce the equations of motion.

**Solution :** Let a particle of mass be moving in a space and be its position at any time .

 The K. E., and P. E.,

Lagrangian,

Lagrange’s equations of motion are given by

 .

Here and .

For , the equation is

 or

Similarly, for , we get

 ,

If the conservative force is then ,

i.e., .

 , , are the required equations.

**Example 2.** Construct the Lagrangian for a particle moving in a plane and then deduce the equation of motion using (i) cartesian coordinates (ii) polar coordinates

**Solution :** **(i) In Cartesian coordinates :**

Let be the position of a particle of mass at time .

Then the K. E., and P. E., of the particle are

 Lagrangian,

Lagrange’s equations of motion are

where ’s are the generalised coordinates.

Here and

For Lagrange’s equation is

 or

Similarly, for

If the conservative force is , then ,

i.e.,

 , are the required equations.

**(ii) In Polar coordinates:**

Let be the position of a particle P at time . Then the K. E., and P. E., of the particle are

 Lagrangian,

Lagrange’s r-equation is

Lagrange’s θ-equation is

Now, ,

i.e.,

 , are the required equations.

**Example 3.** Two particles of masses are connected by a light inextensible string which passes over a small smooth fixed pulley. If , then show that the common acceleration of the particle is .

**Solution :** The system of two particles of masses connected by a light inextensible string passing over a small smooth fixed pulley(Small smooth pulley means pulley is assumed frictionless and massless) is as shown in the figure below.

Let be the length of the string and = distance of from AB.

Then distance of from AB.

Clearly there is only one independent coordinate , which is

the generalised coordinate of the system.

and

Lagrange’s equation is

 . This gives the value of the common acceleration .

**Example 4.**  Construct a Lagrangian for a spherical pendulum and then obtain the Lagrange’s equations of motion.

**Solution :** A spherical pendulum consists of a mass suspended by a rigid weightless rod such that the mass point moves on a surface of a sphere of radius . Here the position of the mass point is determined by three quantities (spherical polar coordinates).

Since is held fixed the system has two degrees of freedom, therefore we say that the mass point P is specified by , where is constant. Hence, the generalised coordinates are θ and ϕ.

P(m)

r

θ

ϕ

O

Y

Z

X

K. E.,

[ ]

P. E.,

 The Lagrangian is

Lagrange’s -equation is . This gives

 ….. (1)

Similarly, Lagrange’s -equation, gives

 constant h (say) ….. (2)

From (1) and (2), we can derive a second order differential equation in :

**Example 5.**  Set up the Lagrangian for a simple pendulum and obtain an equation describing its motion.

**Solution :** Let be the mass of the bob and be the angle made by the string OB of the pendulum with the vertical OA. Let be the length of the string OB.

Here the position of the mass is specified by the angle . Therefore the generalized coordinate is the angle .

K.E. , [ displacement AB = ]

The potential energy of mass (taking a horizontal plane through the lowest point A as a reference level) is given by

O

m

B

A

C

θ

Thus, the Lagrangian is,

Lagrange’s equation is , which gives

 , which is the required equation of motion.

**Example 6.** A bead is sliding on a uniformly rotating wire in a force free space. Show that the acceleration of the bead is , where is the angular velocity of rotation .

**Solution :** Given that, the angular velocity of rotation of the wire is.

r

θ

P(x, y)

Y

X

If the position of the bead at time t is (x, y), then

 ,

 The generalized coordinate is .

Now, K. E.,

As there is no force field, the P. E.,

 The Lagrangian of the system is

Lagrange’s equation gives

**Example 7.** Two mass points of masses and are connected by a string passing through a hole in a smooth table so that rests on the table surface and hangs suspended. Assuming moves only in a vertical line, what are the generalised coordinates for the system? Write down the Lagrangian for the system. Reduce the problem to a single second order differential equation and obtain a first integral of the equation.

**Solution:** Let be the total length of the string. Consider OX as an initial line.

Let be the position of on the table.

 ,

The system is specified by two **generalised coordinates** .

K. E. of ,

K. E. of ,

P. E. of , [ on the table P. E. = 0 ]

P. E. of ,

**Lagrangian** of the system is

Lagrange’s -equation, gives

or ….. (1)

and Lagrange’s θ-equation, gives

 ….. (2)

(1) and (2)

or ….. (3)

This is the required **second order differential equation**.

Multiplying (3) by we get

Integrating w. r. t. , we get

 constant

This is the desired **first integral**.

**Velocity-dependent potentials**

Lagrange’s equations are , j = 1, 2, . . . , N ….. (1)

where

For conservative systems equations (1) can be written in the form

 ….. (2)

where and

Suppose that the given system is not conservative but the generalised forces are obtainable from a function

Define

Then Lagrange’s equations (1) become

or , j = 1, 2, . . . , N. ….. (3)

where

Equations in (2) and (3) are of the same form.

The quantity U is called the **generalised potential** or **velocity dependent potential**.

**Theorem**  The Lagrange’s equations of motion can be written in the form

for a system which is partly conservative. The quantity L refers to the conservative part and to the forces which are not conservative.

**Proof :** In general, Lagrange’s equations are written in the form

 ….. (1)

 where ….. (2)

Suppose that the system is partly conservative. Then,

where and are conservative and non conservative parts of respectively.

By definition of conservative forces, we write where V = P. E.

 ….. (3)

where are the components of generalised forces which are non conservative.

Combining (1) and (3), we get

 [ ]

or , i = 1, 2, …, N

The quantity is the Lagrangian which corresponds to conservative part of the system and are non conservative.

**SOLVED EXAMPLES**

**Example 1.** Show that the Lagrange’s equations can also be written in the form .

**Solution :** We have ,

Differentiating partially w. r. t. , we get

 ….. (1)

Also, [is a fun of

 ….. (2)

(1) and (2)

**Example 2.** If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange’s equations, show by direct substitution that

also satisfies Lagrange’s equations where F is any arbitrary but differentiable function of its argument.

**Solution :** Lagrange’s equations are

 , ….. (1)

Let , where

Then or

 and

 Eqn. (1) becomes

 ….. (2)

Now

 ….. (3)

 [ using (3) ]

 ….. (4)

(2) and (4) ⟹ , which are the Lagrange’s equations for L’.

Thus satisfies Lagrange’s equation.

**Example 3.** For a mechanical system the generalized coordinates appear separately in the kinetic energy and the potential energy such that

 .

Show that the Lagrange’s equations reduce to .

**Solution :** Given that

 and

∴ Lagrange’s equations assume the forms

or .

**Example 4.** A particle moves in a plane under the influence of a force acting towards a centre of force whose magnitude is , where is the distance of the particle to the centre of force. Find the generalized potential that will result in such a force.

**Solution :** If is the generalised potential, then we have

Here the generalized coordinate is .

But given force is

This shows that