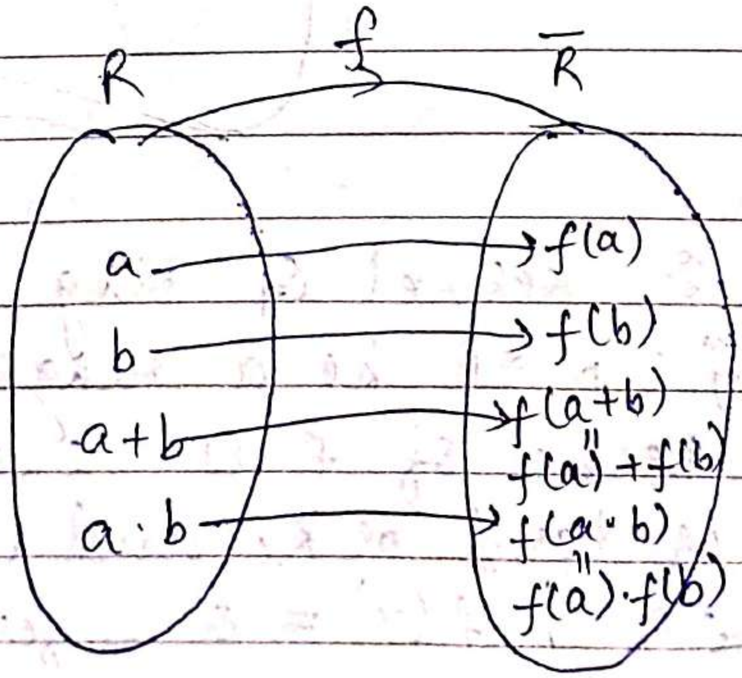
Homomorphism of ring :-

Def^n :- let R & $\bar{R}$ be 2 rings. A mapping $f : R \to \bar{R}$ is called a homomorphism if

(i) $f(a+b) = f(a) + f(b)$

(ii) $f(ab) = f(a) \cdot f(b)$

$\forall a, b \in R$.



REMARKS :-

let $f : R \to \bar{R}$ be a ring homomorphism. Then

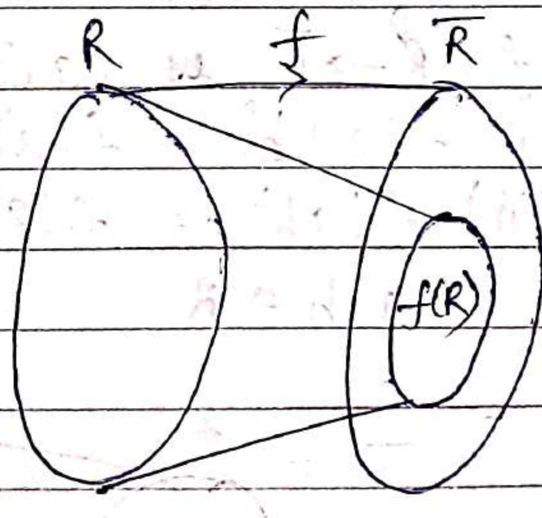
(1) If f is onto then R and $\bar{R}$ are homomorphic rings.

(2) If f is one-one then $f : R \to \bar{R}$ is an isomorphism.

(3) If isomorphism f is onto then R and $\bar{R}$ are isomorphic rings i.e $R \approx \bar{R}$

NOTE :-

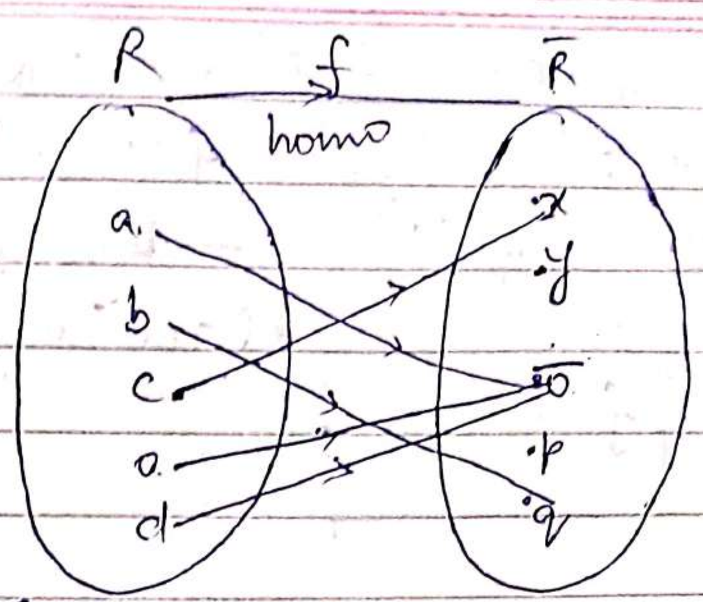
We denote $f(R)$ as homomorphic image of ring R.



Defn :- Kernel of Ring homomorphism :-

let $f : R \to \bar{R}$ be a ring homomorphism then kernel of f is denoted by $I(f)$ or $N(f)$ or $\ker f$ or K_f and is defined by

Thus
$\ker f = \{ x \in R \mid f(x) = \bar{0}$, zero element of $\bar{R} \}$



$\ker f = \{a, 0, d\}$

Theorem(7):-

(8) Let R and $\overline{R}$ be rings with zero element 0 and $\overline{0}$ resp and $f: R \to \overline{R}$ be a homomorphism. Then prove that

(i) $f(0) = \overline{0}$

(ii) $f(-a) = -f(a) \quad \forall a \in R$,

(iii) $f(a-b) = f(a) - f(b) \quad \forall a, b \in R$

Q.12 (iv) $I(f)$ is an ideal of R.

Proof:- Given that R and $\overline{R}$ be rings with zero element 0 and $\overline{0}$ resp.

Also $f: R \to \overline{R}$ is a ring homomorphism

$\therefore$ We have

(1) $f(a+b) = f(a) + f(b)$

(2) $f(a \cdot b) = f(a) \cdot f(b)$

$\forall a, b \in R$.

(i) To prove $f(0) = \overline{0}$

$f(0) = f(0+0)$ — By G_3 of R, in R.

$f(0) = f(0) + f(0)$ — using (i)

$\Rightarrow f(0) + \bar{0} = f(0) + f(0)$ — By G_3 of R_1 in $\bar{R}$

$\Rightarrow \bar{0} = f(0)$ — By left cancellation law in gp $(\bar{R}, +)$

Hence $\boxed{f(0) = \bar{0}}$

(ii) To prove $f(-a) = -f(a)$ $\forall a \in R$

let $a \in R$

Then $a + (-a) = 0$ - By G_4 of R_1 in R.

$\therefore f[a + (-a)] = f(0)$

$\Rightarrow f(a) + f(-a) = \bar{0}$ — By (i) and part(1)

$\therefore f(a) + f(-a) = \bar{0} = f(-a) + f(a)$

↳ By G_5 of R_1 in $\bar{R}$

$\Rightarrow f(-a)$ is additive inverse of $f(a)$

↳ By G_4 of R_1 in $\bar{R}$

i.e $f(-a) = -f(a)$

$\therefore f(-a) = -f(a)$, $\forall a \in R$

(iii) To prove $f(a-b) = f(a) - f(b)$ $\forall a, b \in R$

let $a, b \in R$

Then $a - b = a + (-b) \in R$

$\therefore f(a-b) = f[a + (-b)]$.

$= f(a) + f(-b)$ — by (1)

$= f(a) + [-f(b)]$ - By part(2).

$= f(a) - f(b)$

$\therefore f(a-b) = f(a) - f(b), \forall a, b \in R$

(iv) To prove $I(f)$ is an ideal of R.

We have $f(0) = \bar{0}$

and $I(f) = \{x \in R \mid f(x) = \bar{0}\}$.

$\Rightarrow 0 \in I(f)$

$\therefore I(f)$ is a non-empty subset of R.

let $x, y \in I(f) \subseteq R$.

$\Rightarrow f(x) = \bar{0}$ and $f(y) = \bar{0}$ – By def^n.

Now $f(x-y) = f(x) - f(y)$ – By part(3)

$$\begin{aligned} f(x-y) &= \bar{0} - \bar{0} \\ &= \bar{0} + (-\bar{0}) \\ &= \bar{0} + \bar{0} \\ &= \bar{0} \end{aligned}$$

$\therefore f(x-y) = \bar{0}$

$\Rightarrow x - y \in I(f)$ – By def^n.

$\therefore x - y \in I(f) \quad \forall x, y \in I(f)$

$\Rightarrow I(f)$ is a subgroup of R under $+$ – (A)

Also, let $r \in R$ and $x \in I(f) \subseteq R$

$\Rightarrow f(x) = \bar{0}$

$rx, xr \in R$ – By G_1 of R_2 in R.

$$\begin{aligned} \therefore f(rx) &= f(r) \cdot f(x) \quad \text{– By (2)} \\ &= f(r) \cdot \bar{0} \\ &= \bar{0} \end{aligned}$$

↳ By theorem (1) part (1) in $\bar{R}$

$\Rightarrow rx \in I(f)$

similarly $xr \in R$ for $x, r \in R$

$\Rightarrow f(xr) = f(x) \cdot f(r)$

$= \bar{0} \cdot f(r)$

$f(xr) = \bar{0}$

$\Rightarrow xr \in I(f)$

$\therefore rx, xr \in I(f) \ \forall r \in R, \forall x \in I(f)$ — (B)

Using (A) & (B) $I(f)$ is an ideal of R.

REMARK :-

We proved that $f(0) = \bar{0}$. However this may not happens for the unit element.

For eg:- Consider $f : R \to \bar{R}$ defined as $f(x) = \bar{0} \ \forall x \in R$ where $\bar{0}$ is zero element of $\bar{R}$.

Let $x, y \in R$. Then $x + y \in R$ — By G_1 of R_1 in R

Also $x \cdot y \in R$ — By G_1 of R_2 in R

$\therefore f(x+y) = \bar{0} = \bar{0} + \bar{0} = f(x) + f(y)$

$f(xy) = \bar{0} = \bar{0} \cdot \bar{0} = f(x) \cdot f(y)$

$\Rightarrow f : R \to \bar{R}$ is a ring homomorphism

If $1 \in R$ and $\bar{1} \in \bar{R}$ then

$f(1) = \bar{0}$ — By (1)

$\neq \bar{1}$

Thus $f(1) = \bar{1}$ is not always true

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Such type of homomorphism is called zero homomorphism.

NOTE :-
For the zero homomorphism
$\ker f = \{x \in R \mid f(x) = \bar{0}\} = R$

Eg :-
let R be a ring with unit element 1 and f be a homomorphism of R into a ring R'
If $f(R) = R'$ or R' is an integral domain with $\ker f \neq R$, then prove that $f(1)$ is the unit element of R'?

Proof :- Given that $f : R \to R'$ is a ring homomorphism where R is a ring with unity 1.
Case I :-
let $f(R) = R'$
$\Rightarrow$ f is onto homomorphism
$\therefore$ every element of R' is an image of some element of R.
let $x' \in R'$ then $\exists$ some $x \in R$ s.t $f(x) = x'$ — (1)
We have $1 \in R$
$\therefore x \cdot 1 = x = 1 \cdot x \quad \forall x \in R$ — By def^n of unity.

$\Rightarrow f(x \cdot 1) = f(x) = f(1 \cdot x)$

$\Rightarrow f(x) \cdot f(1) = f(x) = f(1) \cdot f(x)$ $\left\{ \because f \text{ is homomorphism} \right.$

i.e $x' \cdot f(1) = x' = f(1) \cdot x'$ — By (1)

$\therefore x' \cdot f(1) = x' = f(1) \cdot x'$ — $\forall x' \in R'$

$\Rightarrow f(1)$ is a unit element of R'

i.e $f(1) = 1'$

Case II :-

let R' be an integral domain an

with kernel $f \neq R$

i.e $\{x \in R \mid f(x) = 0'\} \neq R$

where $0'$ is zero element of R'

But $0 \in \ker f$ $\because f(0) = 0'$

$\Rightarrow \ker f \subset R$ (proper subset)

$\therefore \exists\, a \in R,\ a \neq 0$ s.t

$f(a) \neq 0'$

$\Rightarrow a \notin \ker f$.

let $f(a) = a' \neq 0'$ in R' — (2)

Now $a \cdot 1 = a = 1 \cdot a \quad \forall a \in R$.

$\Rightarrow f(a \cdot 1) = f(a) = f(1 \cdot a)$

$\Rightarrow f(a) \cdot f(1) = f(a) = f(1) \cdot f(a)$ $\because f$ is homo — (3)

Now let $b' \in R'$ Rem

Then $a'b' = f(a)\,b'$ — by (2)

$= f(a)\cdot f(1)\cdot b'$ — By (3)

$a'b' = a'\cdot f(1)\cdot b'$ — By (2)

$\Rightarrow b' = f(1)\cdot b'$ — By cancellation law in integral domain R'.

Simillarly,

$b'a' = b' f(a)$ — By (2)

$= b'\cdot f(1)\cdot f(a)$ — By (3)

$b'a' = b'\cdot f(1)\cdot a'$ — By (2)

$\Rightarrow b' = b' f(1)$ — By right cancellaⁿ law in integral domain R'.

Thus,

$b' f(1) = b' = f(1)\, b' \quad \forall b' \in R'$

$\Rightarrow f(1)$ is unity of R'

i.e $f(1) = 1'$

[W8][W5] Theorem :-

Q.(13) Prove that a homomorphism f of a ring R to a ring R' is an isomorphism iff $\ker f = \{0\}$?

Proof :- Given that $f : R \to R'$ is a ring homomorphism.

Then by a result we have $f(0) = 0'$

Where 0 & $0'$ are zero element of R & R' resp.

Also $\ker f = \{x \in R \mid f(x) = 0'\}$

Part I :-

Suppose that f is an isomorphism.

$\Rightarrow$ f is homomorphism and one-one

we want to prove that $\ker f = \{0\}$

let $x \in \ker f$.

$\therefore$ By def^n, $f(x) = 0'$

$\therefore$ $f(x) = f(0)$

But f is 1-1.

$\Rightarrow$ $x = 0$.

Hence $\ker f = \{0\}$.

Part II :-

Suppose that $\ker f = \{0\}$

we want to show that f is 1-1.

let $x, y \in R$ with $f(x) = f(y)$.

$\Rightarrow$ $x, -y \in R$ — By G_4 of R_1 in R

$\Rightarrow$ $x + (-y) \in R$ — By G_1 of R_1 in R

$\therefore$ $f(x + (-y)) = f(x) + f(-y)$ $\because f$ is homo

$= f(x) + [-f(y)]$ — By a result

$= f(y) + [-f(y)]$ — By hypothesis

$= 0'$ — By G_4 of R_1 in R'

Thus $f(x-y) = 0'$

$\Rightarrow$ $x - y \in \ker f$ — By def^n

$\Rightarrow$ $x - y = 0$ — By data

$\Rightarrow$ $x = y$

Hence $f(x) = f(y) \Rightarrow x = y$

$\therefore$ f is one-one.

Q(14) let $J(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in Z\}$ is an integral domain under usual addition and multipⁿ. Define $f: J(\sqrt{2}) \to J(\sqrt{2})$ by $f(a+b\sqrt{2}) = a - b\sqrt{2}$. Prove that f is an onto isomorphism?

Proof:- Given that

$J(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in Z\}$

is an integral domain under usual addⁿ and multipⁿ.

Define a relaⁿ $f: J(\sqrt{2}) \to J(\sqrt{2})$ by

$f(a+b\sqrt{2}) = a - b\sqrt{2}$ — (1)

(i) To show f is homo:-

let $x, y \in J(\sqrt{2})$ domain

$\Rightarrow x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$ for some

$a, b, c, d \in Z$

$x + y = (a + b\sqrt{2}) + (c + d\sqrt{2})$

$x + y = (a + c) + (b + d)\sqrt{2}$ — (2)

$a, c \in Z \Rightarrow a + c \in Z$

$b, d \in Z \Rightarrow b + d \in Z$

$\therefore (a+c) + (b+d)\sqrt{2} \in J\sqrt{2}$

i.e $x + y \in J\sqrt{2}$

$\therefore f(x+y) = f((a+c) + (b+d)\sqrt{2})$ — By (2)

$= (a+c) - (b+d)\sqrt{2}$ — using (1)

$= (a - b\sqrt{2}) + (c - d\sqrt{2})$

$= f(a+b\sqrt{2}) + f(c+d\sqrt{2})$ — By (1)

$= f(x) + f(y)$

$\therefore f(x+y) = f(x) + f(y) \quad \forall x, y \in J(\sqrt{2})$

Altr show $f(x \cdot y) = f(x) \cdot f(y) \ \forall x, y \in J(\sqrt{2})$

$\Rightarrow$ mapping f is a homo

(ii) To show f is 1-1

let $f(x) = f(y)$ for $x, y \in J(\sqrt{2})$

$\Rightarrow f(a+b\sqrt{2}) = f(c+d\sqrt{2})$

$\Rightarrow a - b\sqrt{2} = c - d\sqrt{2}$ — By (1)

$\Rightarrow (a-b\sqrt{2}) - (c-d\sqrt{2}) = 0$

$\Rightarrow (a-c) + (d-b)\sqrt{2} = 0$

$\Rightarrow a-c=0$ and $(d-b)\sqrt{2} = 0$

$\Rightarrow a = c$ & $d-b=0 \quad \because \sqrt{2} \neq 0$

$\therefore$ We have $a = c$ & $d = b$

So $x = a+b\sqrt{2}$

$= c+d\sqrt{2}$

$= y$

Thus $f(x) = f(y) \Rightarrow x = y$

$\therefore f$ is 1-1 mapping.

(iii) To show f is onto

let $p+q\sqrt{2} \in J(\sqrt{2})$, codomain

Then there exists an element

$p-q\sqrt{2} \in J(\sqrt{2})$, domain

s.t $f(p-q\sqrt{2}) = p+q\sqrt{2}$

$\Rightarrow p+q\sqrt{2}$ is an image

Hence every element of codomain $J(\sqrt{2})$ is an image

$\Rightarrow$ f is onto
Thus f is homo and one-one
$\Rightarrow$ f is isomorphism
Also f is onto
Hence proved.

Q(15) If U is an ideal of ring R. Then prove that R/U is a homomorphic image of R.

Proof:- Given that U is an ideal of a ring R
Then R/U is the quotient ring under coset add^n and multiplication defined as

$$(U+x)+(U+y) = U+(x+y) \quad \text{—} \; (1)$$
$$(U+x)\cdot(U+y) = U+(x\cdot y) \quad \text{—} \; (2)$$

$\forall\, x, y \in R$.

And zero element of R/U is $U+0 = U$
Where 0 is zero element of R.
We define a mapping $\phi : R \to R/U$ as

$$\phi(x) = U+x \quad \forall\, x \in R \quad \text{—} \; (3)$$

(i) To prove ϕ is homomorphism.
let $x, y \in R$ Then $x+y \in R$

$$\therefore \phi(x+y) = U+(x+y) \quad \text{- using (3)}$$
$$= (U+x)+(U+y) \quad \text{- By (1)}$$
$$= \phi(x)+\phi(y) \quad \text{- using (3)}$$
$$\therefore \phi(x+y) = \phi(x)+\phi(y) \quad \forall\, x, y \in R$$

$\Rightarrow$ ϕ is homomorphism.

(ii) To show ϕ is onto.

let $U+a \in R/U$. Then there is $a \in R$ s.t (for some $a \in R$)

$\phi(a) = U + a$.

$\Rightarrow$ $U+a$ is a image

Hence every element of R/U is an image

$\Rightarrow$ ϕ is onto

Thus $\phi : R \to R/U$ is a homomorphism and onto.

$\therefore$ R/U is a homomorphic image of R.

i.e $\phi(R) = R/U$.

[W6]

Q.(16) If $f(x)$ is a polynomial over a field F, then prove that $x-\alpha$ divides $f(x)$ if and only if $f(\alpha) = 0$?

Sol^n:- let $f(x)$ be a polynomial over a field F.

let us divide $f(x)$ by $x-\alpha$.

Then by divison algorithm $\exists$ quotient $q(x)$ and remainder $f(\alpha)$ s.t

$$f(x) = (x-\alpha)\, q(x) + f(\alpha) \quad \text{--- (1)}$$

Suppose that $x-\alpha$ divides $f(x)$

$\Rightarrow$ Remainder $f(\alpha) = 0$

Conversely suppose that $f(\alpha) = 0$

(1) $\Rightarrow$ $f(x) = (x-\alpha)\, q(x)$

$\Rightarrow$ $(x-\alpha)$ divides $f(x)$

Hence proved.